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LONGEST PATHS IN DIGRAPHS

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In this paper, we give a sufficient condition on the degrees of the vertices of a digraph to insure the existence of a path of given length, and we characterize the extremal graphs.

Notations

We use standard terminology ([1] or [4]). A digraph (1-graph) $D=(X, U)$ consists of a finite set of vertices X and a set U of ordered pairs (x, y) of vertices called arcs. In what follows the digraphs considered are without loops or multiple arcs. We denote by:

$$\Gamma^+(x) = \{y|y\in X,\ (x,y)\in U\},\quad d^+(x) = |\Gamma^+(x)|\quad \text{(outdegree)}$$

$$\Gamma^-(x) = \{y|y\in X,\ (y,x)\in U\},\quad d^-(x) = |\Gamma^-(x)|\quad \text{(indegree)}$$

$$d(x) = d^+(x)+d^-(x)\quad \text{(degree)}$$

For $A\subseteqq X$, $d_A^+(x)=|\Gamma^+(x)\cap A|$; $d_A^-(x)=|\Gamma^-(x)\cap A|$.

When we speak of paths (circuits) in digraphs, we always mean directed path (circuits). The length of a path (circuit) is the number of arcs of this path (circuit). A digraph is strong if for any two vertices x and y there exists a path from x to y and a path from y to x. A digraph is connected if the underlying graph is connected.

Introduction

The aim of the article is to obtain sufficient conditions on the degrees of the vertices of a digraph in order to insure the existence of a path of length k (or equivalently $\geqq k$). For other results concerning sufficient conditions on the number of arcs or circuits see [2, 4, 6, 7, 10]. First let us recall two results in the undirected case due to Dirac (stronger results have been obtained by many authors (see [3])).

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Theorem 1 (Dirac). *If G is a connected graph of minimum degree at least k, then G contains a path of length* $\min(2k, n-1)$.

This theorem is best possible and follows from an analogous theorem on cycles.

Theorem 1′. *If G is a 2-connected graph of minimum degree at least k, then G contains a cycle of length greater than or equal to* $\min(2k, n)$.

Theorems 1 and 1′ cannot be extended to digraphs by considering only the (total) degree. Indeed Ghouila-Houri [5] has shown that for any k, there exist strong digraphs with minimum degree at least k and without paths of length 8. In this case the best possible result is:

Theorem 2. *If a digraph D of order n has the property that for any two non-adjacent vertices x and y $d(x)+d(y) \geqq 2n-2h-1$, with $0<h<n$, then D contains a path of length $\left\lceil \frac{n}{h} \right\rceil - 1$. ($\lceil x \rceil$ means the smallest integer greater than or equal to x).*

This theorem is in fact a corollary of results of Las Vergnas [9]. It is also an easy consequence of Heydemann's results [6, 7].

Theorem 2′. [6, 7]. *Let D be a strong digraph of order n such that for any pair of non-adjacent vertices x and y, $d(x)+d(y) \geqq 2n-2h+1$, then D contains a circuit of length greater than or equal to $\left\lceil \frac{n-1}{h} \right\rceil + 1$.*

However we can impose conditions on the outdegrees or indegrees. A possible generalisation of theorem 1′ was the following conjecture made some years ago by Bermond and Thomassen (see [4]): let D be a strongly 2-connected digraph of order n and minimum indegree and outdegree at least k, then D contains a circuit of length greater than or equal to $\min(2k, n)$. Recently Thomassen (see [4 or 10] for more details) showed that this conjecture was false by giving examples of strongly 2-connected digraphs with $d^+(x) \geqq k$, $d^-(x) \geqq k$ for every vertex x and where the longest circuits have length $k+2$. Another possible generalization of theorem 1′ is the following conjecture of Thomassen [10].

Conjecture ([10]). *If a digraph D has minimum indegree and outdegree at least k and if any two vertices of D are on a common circuit, then D contains a circuit of length greater than or equal to* $\min(2k, n)$.

This conjecture, if true, would imply our main theorem (in case $h=k$) which extends theorem 1.

Main Theorem. *Let D be a connected digraph of order n such that for every vertex x, $d^+(x) \geqq k$ and $d^-(x) \geqq h$, then*

A: *D contains a path with* $\min(n, h+k+1)$ *vertices.*

B: *If $n>h+k+1$, the only digraphs with no path with $h+k+2$ vertices are the union of K^*_{k+1} having exactly one vertex in common. (Note that this implies in particular that $h=k$ and $n-1$ is a multiple of k).*

(K^*_{k+1} denotes the complete symmetric digraph on $k+1$ vertices.)

Proofs

Proof of A. The proof given here follows ideas of C. Thomassen and is much more shorter than the original proof we had.

First, suppose D is strong. If $k+h+1 \geqq n$, the theorem is a consequence of Ghouila-Houri's theorem [5] on hamiltonian paths. Thus we will suppose $n > h+k+1$.

Let C be a longest circuit in D and c its length. Clearly we have

(1) $$c \geqq \max(h, k)+1.$$

If C is hamiltonian, A is proved. Thus let us assume $c<n$.

Let P be a longest path in $D-C$ (subgraph generated by the vertices of D not belonging to C) with p vertices, $p \geqq 1$, say $P=x_1, x_2, \ldots, x_p$. Let s be the number of arcs from C to x_1 $\left(s=d_C^-(x_1)\right)$ and r be the number of arcs from x_p to C $\left(r=d_C^+(x_p)\right)$. From the maximality of P we have

$$d^-(x_1) = s+d_P^+(x_1) \quad \text{and} \quad d^+(x_p) = r+d_P^+(x_p),$$

so

(2) $$p \geqq h-s+1$$ $$p \geqq k-r+1$$

and then

(3) $$r+s \geqq h+k+2-2p.$$

If $r=0$ (resp. $s=0$) then $\Gamma^+(x_p) \subset V(P)$ $\left(\text{resp. } \Gamma^-(x_1) \subset V(P)\right)$ and D contains an other circuit C', vertex-disjoint from C, with c' vertices where $c' \geqq k+1$ (resp. $c' \geqq h+1$). In both cases, as D is strong, there is a path between C and C' so that we get a path with $c+c'$ vertices where

$$c+c' \geqq \max(h, k)+\min(h, k)+2 = h+k+2.$$

It remains to consider the case $r \geqq 1$, $s \geqq 1$. From the maximality of C, if there exists an arc (c_i, x_1) then there is no arc (x_p, c_{i+j}) for $1 \leqq j \leqq p$ and thus we have:

(4) $$r+s \leqq c-p+1.$$

Since $r \geqq 1$, we get a path with $c+p$ vertices by (4)

$$c+p \geqq r+s+2p-1$$

and by (3)

(5) $$c+p \geqq h+k+1$$

If D is connected, but not strong let us consider the strong components of D. Let a component be called component source (resp. sink) if every vertex of this component is not the end (resp. origin) of an arc with origin (resp. end) in an other component. As D is connected D contains at least a component source and a component sink with a directed path between the two components. In the component source $d^-(x) \geqq h$ and there exists therefore a circuit of length $\geqq h+1$ in it. Similarly, there exists a circuit of length $k+1$ in the component sink. With the path between we obtain a path with $h+k+2$ vertices. ∎

Proof of B. From the proof of A, if D does not contain a path with $h+k+2$ vertices, we have $r\geqq 1$, $s\geqq 1$ and equalities in (2), (3), (4), (5). That is

$$(6)\qquad \begin{aligned} r &= k-p+1 \\ s &= h-p+1 \\ r+s &= h+k-2p+2 = c-p+1 \\ c+p &= h+k+1 \end{aligned}$$

So $d_P^+(x_p)=p-1$ $d_P^-(x_1)=p-1$, and in particular there is an arc (x_p, x_1) and therefore a circuit $C'=(x_1, \ldots, x_p, x_1)$ in $D-C$.

B.1. — The subdigraph induced by $\{x_1, x_2, \ldots, x_p\}$ is a complete symmetric digraph. Suppose there is no arc (x_i, x_j) for some i, j; consider the path P' in $D-C$: $P'=x_j, x_{j+1}, \ldots, x_p, x_1, \ldots, x_{j-1}$. By the maximality of P and (6) $d_C^-(x_j)=$ $=s'\geqq s+1$ and $d_C^+(x_{j-1})=r'\geqq r$. But the proof of A with P' instead of P gives a path with $c+p\geqq r'+s'+2p-1\geqq h+k+2$ vertices.

B.2. — Structure of the subdigraph induced by the vertices of $P\cup C$.
As the subdigraph induced by the vertices of P is complete and as by (6) and the maximality of P, $d_C^+(x_i)=r$, $d_C^-(x_i)=s$, we can label the vertices of C in a consecutive order $v_1, \ldots, v_c$ such that the arcs between P and C are

$$(x_i, v_j), \quad \text{for} \quad 1 \leqq i \leqq p, \quad \text{for} \quad p+1 \leqq j \leqq p+r$$

and

$$(v_j, x_i), \quad \text{for} \quad 1 \leqq i \leqq p, \quad \text{for} \quad p+r \leqq j \leqq c.$$

Now the same arguments applied with $C_1=(x_1, \ldots, x_p, v_{p+1}, \ldots, v_c, x_1)$ instead of C and $P_1=v_1, \ldots, v_p$ instead of P show that the subdigraph induced by $\{v_1, \ldots, v_p\}$ is complete symmetric and that the arcs between P_1 an $P\cup(C-P_1)$ are the arcs

$$(v_i, v_j) \quad \text{for} \quad 1 \leqq i \leqq p; \quad p+1 \leqq j \leqq p+r$$

$$(v_j, v_i) \quad \text{for} \quad 1 \leqq i \leqq p; \quad p+r \leqq j \leqq c.$$

B.3. — $r=1$.
Suppose $r>1$. Let $p+1\leqq j\leqq p+r-1$. By *B.2*, there is no arc (v_j, v) for $v\in P$ or $v\in\{v_1, \ldots, v_p\}$. There is no arc (v_j, v) with $v\in D-P-C$ otherwise the path $v_{p+r+1}, \ldots, v_c, v_1, \ldots, v_p, v_{j+1}, \ldots, v_{p+r}, x_1, \ldots, x_p, v_{p+1}, \ldots, v_j, v$ would have $h+k+2$ vertices. There is also no arc $(v_j, v_{j'})$ with $p+r+1\leqq j'\leqq c$ otherwise the circuit $(v_j, v_{j'}, v_{j'+1}, \ldots, v_c, v_1\ldots, v_p, v_{j+1}, \ldots, v_{j'-1}, x_1, \ldots, x_p, v_{p+1}, \ldots, v_j)$ would be of length $c+p$ contradicting the maximality of C. Thus $d^+(v_j)\leqq r-1<k$ a contradiction.

B.4. — Similarly $s=1$ and thus $p=h=k$ by (6).

B.5. — Therefore the subdigraph induced by $P\cup C$ consists of two complete symmetric digraphs on k vertices K_1 and K_2 and a vertex x joined in the two directions to all the vertices of $K_1\cup K_2$. Furthermore all the arcs between $P\cup C$ and the rest of the digraph have x as end vertex.

B.6. — End of the proof.
By B.4., $h=k$ and by B.5. $C\cup P$ has $2k+1$ vertices. Let v be a vertex of $D-(P\cup C)$, as by B.5. it is joined at most to x in $P\cup C$, its outdegree in $D-(P\cup C)$ is at least $k-1$. Thus the subdigraph $D-(P\cup C)$ has at least k vertices and, if

it has k vertices, it is a complete symmetric digraph, and all its vertices are joined in the two directions to x. Therefore $n \geqq 3k+1$ and if $n=3k+1$, D consists of three K_{k+1}^* having exactly one vertex in common. Theorem B is proved for $2k+1 < n \leqq 3k+1$. Suppose B is proved for $n \leqq qk+1$ and let $qk+1 < n \leqq (q+1)k+1$ (with $q \geqq 3$). The digraph $D-K_1$ satisfies the hypothesis of theorem B and has $n-k$ vertices, where $2k+1 < n-k \leqq qk+1$. By induction hypothesis $D-K_1$ has the structure given in B and D also. ∎

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